THE YIELD-LINE THEORY FOR CONCRETE SLABS

by

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THE YIELD-LINE THEORY FOR CONCRETE SLABS

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SYNOPSIS

An outline of the yield-line theory, a plastic theory for the prediction of ultimate flexural strength of reinforced concrete slabs, developed by K. W. Johansen, is presented. The theory is based on plastic behavior occurring in a pattern of yield-lines, the location of which depends on loading and boundary conditions. The ultimate flexural strength may be evaluated, even for complex slabs, with limited mathematical effort. The theoretical strengths obtained are in good agreement with experimental results and generally on the safe side thereof. The use of the theory is illustrated by a numerical example.

INTRODUCTION

To carry loads safely is the primary function of most reinforced concrete structures. It seems proper, therefore, to base the design of such structures primarily on their ultimate load-carrying capacity. In many cases it is necessary that structures possess stiffness as well as strength to perform the function intended in design. It is then desirable to supplement a design based on strength by considerations of deflections and deformations at working load.

The pioneers of reinforced concrete design used this type of design philosophy (1).[1] They were primarily concerned with the strength of structures, relatively little attention being devoted to conditions at working loads. In later years, however, this design basis was almost reversed by direct applications of the theory of elasticity. The elastic theory is a powerful tool for evaluating stresses and deformations at the working load level, but it is unsatisfactory for estimating ultimate strength of many reinforced concrete structures and structural elements (2). Hence a re-emphasis of strength resulted, and the conditions at working loads were given primary consideration.

Recently, a gradual re-emphasis of strength has taken place in the field of reinforced concrete design. In most countries such a return to original thought is found in the methods used to proportion sections, while the use of the elastic theory is

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[1]Numbers in parentheses refer to references listed in the Bibliography.

continued for the purpose of analyzing indeterminate structures. Examples can be found of the code permitting analyses of statically indeterminate structures with the aid of a plastic theory, whereby the magnitudes of the sectional forces are derived considering the plastic properties of the materials (3).

The basic assumption of the yield-line theory, first developed by Johansen, is that a reinforced concrete slab, similar to a continuous beam or frame of a perfectly plastic material, will develop yield hinges under overload, but will not collapse until a mechanism is formed (4). The hinges in the slab must be long lines, along which the maximum moment the slab can resist will tend to oppose rotation. The general crack pattern which these yield lines will form may sometimes be deduced logically, from geometry, sometimes inferred intuitively, and must sometimes be obtained from model or full-scale tests. Once the general pattern is known, a specific crack pattern may be calculated for a particular support and loading condition using energy and/or force equilibrium. Frequently the resulting equations are too complex for direct solution, and a system of successive approximations must be used.

All these methods provide upper-bound solutions, and it will be necessary to investigate all possible yield line patterns to find the least value of the ultimate load, so our aim is to find the "lowest" value of the upper-bound solutions.

EQUILIBRIUM EQUATIONS METHOD

This method is based on the equilibrium of the edge couples and shears acting on the segments of the slab formed by yield lines (5).

Let us first consider a rectangular slab reinforced in two directions perpendicular to each other, subject to a uniform load w, fixed on all four edges as in Fig. 1.

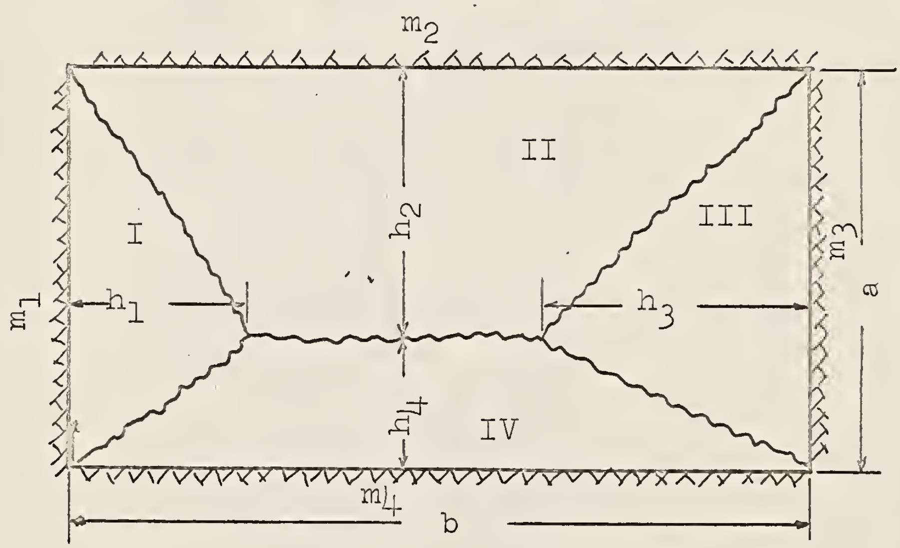


Fig. 1. Rectangular slab.

In the theory of elasticity, a fixed edge calls for certain geometrical boundary conditions. In the yield-line theory, however, the moments m_1, m_2, m_3, and m_4 at a fixed support depend primarily on the amount of negative reinforcement provided. Likewise, m is the positive moment that depends on the amount of uniform positive reinforcement provided in the slab. If such a rectangular slab is overloaded, yielding will begin in the region of high amount and, as loading continues, yield line cracks will form and spread into a pattern referred to as a yield-line

pattern. The load-carrying capacity of the slab will be exhausted when the yield-line cracks have spread to the slab edges, at which load the slab reaches a state of neutral equilibrium (6). (Same as a mechanism formed in plastic design theory. See reference 7.)

The yield-line cracks divide the slab into several parts, and a heavy concentration of curvature takes place at these cracks, exceeding the elastic strength. Near the ultimate load it is assumed that the individual slab parts are plane, all deformations taking place in the yield lines. It then follows that the yield line must be straight, and the deformations of the slab may be considered as rotations of the slab parts about axes in their support. Furthermore, a yield line between two slab parts must pass through the intersection of the axes of rotation of the two parts. Figure 2 shows some typical yield-line patterns for various types of slabs; an axis of rotation must lie in a line of support and must pass through columns. In this manner the general nature of the possible yield-line patterns may be determined (8).

Final determination of the yield-line pattern corresponding to the ultimate load of a slab may be made with the aid of equilibrium conditions for the individual slab parts. The shearing forces acting in the yield lines must then be found. However, since the yield moments are principal moments, twisting moments are zero in the yield lines, and in most cases the shearing forces also are zero (9). Thus only the moment m per unit length of yield line acts perpendicularly to these lines, and the total

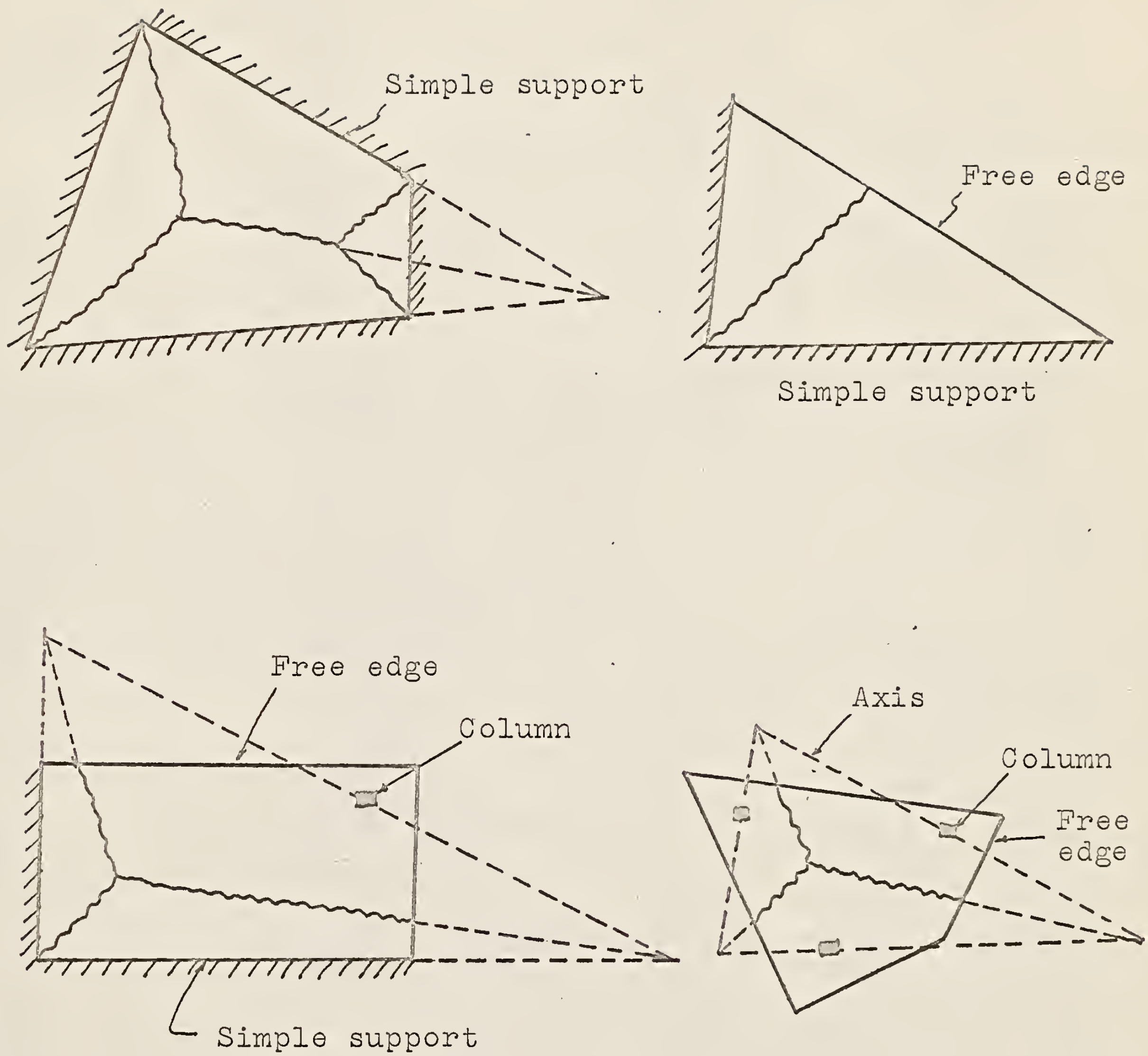


Fig. 2. Typical yield-line crack patterns.

moment may be represented by a vector in the direction of the yield line with magnitude m times the length of the line. The resulting moment for an individual slab part is then found by vector addition. Now let

$$\frac{m_1}{m} = k_1\ ,\quad \frac{m_2}{m} = k_2\ ,\quad \frac{m_3}{m} = k_3\ ,\quad \frac{m_4}{m} = k_4$$

i.e., $m_1 = k_1 m$, $m_2 = k_2 m$, $m_3 = k_3 m$, $m_4 = k_4 m$. The yield line should be formed as indicated in Fig. 1, and equilibrium of the four slab parts gives:

Part I,

$$a(m + m_1) = \frac{1}{2}\, a h_1 w\, \frac{h_1}{3}$$

or

$$am(1 + k_1) = \frac{1}{6} a h_1^{\,2} w \tag{1}$$

Part II,

$$bm(1 + k_2) = \frac{1}{6}\, w h_1 h_2^{\,2} + \frac{1}{6}\, w h_3 h_2^{\,2} + \frac{1}{2}\, w(b - h_1 - h_3) h_2^{\,2} \tag{2}$$

Part III,

$$am(1 + k_3) = \frac{1}{6}\, w a h_3^{\,2} \tag{3}$$

Part IV,

$$bm(1 + k_4) = \frac{1}{6}\, w h_1 h_4^{\,2} + \frac{1}{6}\, w h_3 h_4^{\,2} + \frac{1}{2}\, w(b - h_1 - h_3) h_4^{\,2} \tag{4}$$

and

$$h_2 + h_4 = a \tag{5}$$

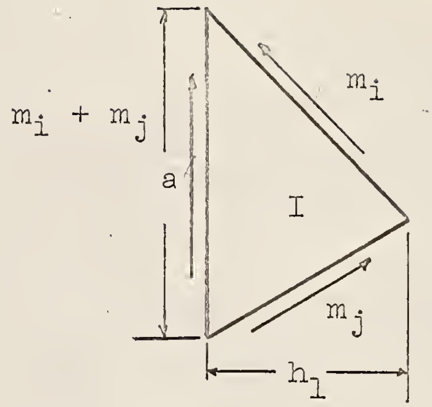


Fig. 3. Vector diagram.

From these five equations it is possible to determine five unknowns m, h_1, h_2, h_3, and h_4 as functions of a, b, k_1, k_2, k_3, and k_4; it is found that

$$m = \frac{wa_r^2}{24}\left[\sqrt{3 + \left(\frac{a_r}{b_r}\right)^2} - \frac{a_r}{b_r}\right]^2 \tag{5}$$

where

$$a_r = \frac{2a}{\sqrt{1 + k_2} + \sqrt{1 + k_4}} \tag{6}$$

$$b_r = \frac{2b}{\sqrt{1 + k_1} + \sqrt{1 + k_3}} \tag{7}$$

These are general equations for rectangular slabs. For some special cases:

(1) For simply supported slabs

$$k_1 = k_2 = k_3 = k_4 = 0$$

and $\quad a_r = a, \quad b_r = b.$

Equation (6) will be

$$m = \frac{wa^2}{24}\left[\sqrt{3 + (a/b)^2} - a/b\right]^2 \tag{9}$$

(2) For rectangular slabs with restrained ends,

$$m_1 = m_2 = m_3 = m_4 = m$$

i.e., $$k_1 = k_2 = k_3 = k_4 = 1$$

$$a_r = a/\ \ 2$$

$$b_r = b/\ \ 2$$

$$m = \frac{wa^2}{48}\left[\sqrt{3 + (a/b)^2} - a/b\right]^2 \tag{10}$$

(3) For square slabs with restrained edges,

$$m_1 = m_2 = m_3 = m_4 = m$$

i.e., $$k_1 = k_2 = k_3 = k_4 = 1$$

and $$a = b, \quad a_r = b_r.$$

$$m = \frac{wa^2}{48}\left[\sqrt{3 + 1} - 1\right]^2 = \frac{wa^2}{48} \tag{11}$$

(4) For simply supported square slabs,

$$m_1 = m_2 = m_3 = m_4 = 0$$

i.e., $$k_1 = k_2 = k_3 = k_4 = 0$$

$$m = \frac{wa^2}{24} \tag{12}$$

(5) For simply supported one-way slabs

$$b_r \longrightarrow \infty$$

$$k_2 = k_4 = 0$$

$$a_r = a$$

$$m = \frac{wa^2}{24} \cdot 3 = \frac{wa^2}{8} \tag{13}$$

(6) For fixed ended one-way slabs with $m_2 = m_4 = m$,

$$b_r \longrightarrow \infty$$

$$k_2 = k_4 = 1$$

$$a_r = a/\sqrt{2}$$

$$m = \frac{wa^2}{2 \times 24} \cdot 3 = \frac{wa^2}{16} \qquad (14)$$

For design purposes Table 1 and the curves in Plate I have been made for any case when $m_1 = m_2 = m_3 = m_4$ (i.e., $k_1 = k_2 = k_3 = k_4$) and the slab is subject to a uniformly distributed load of w.

Table 1. Values of wa^2/m for rectangular slabs.

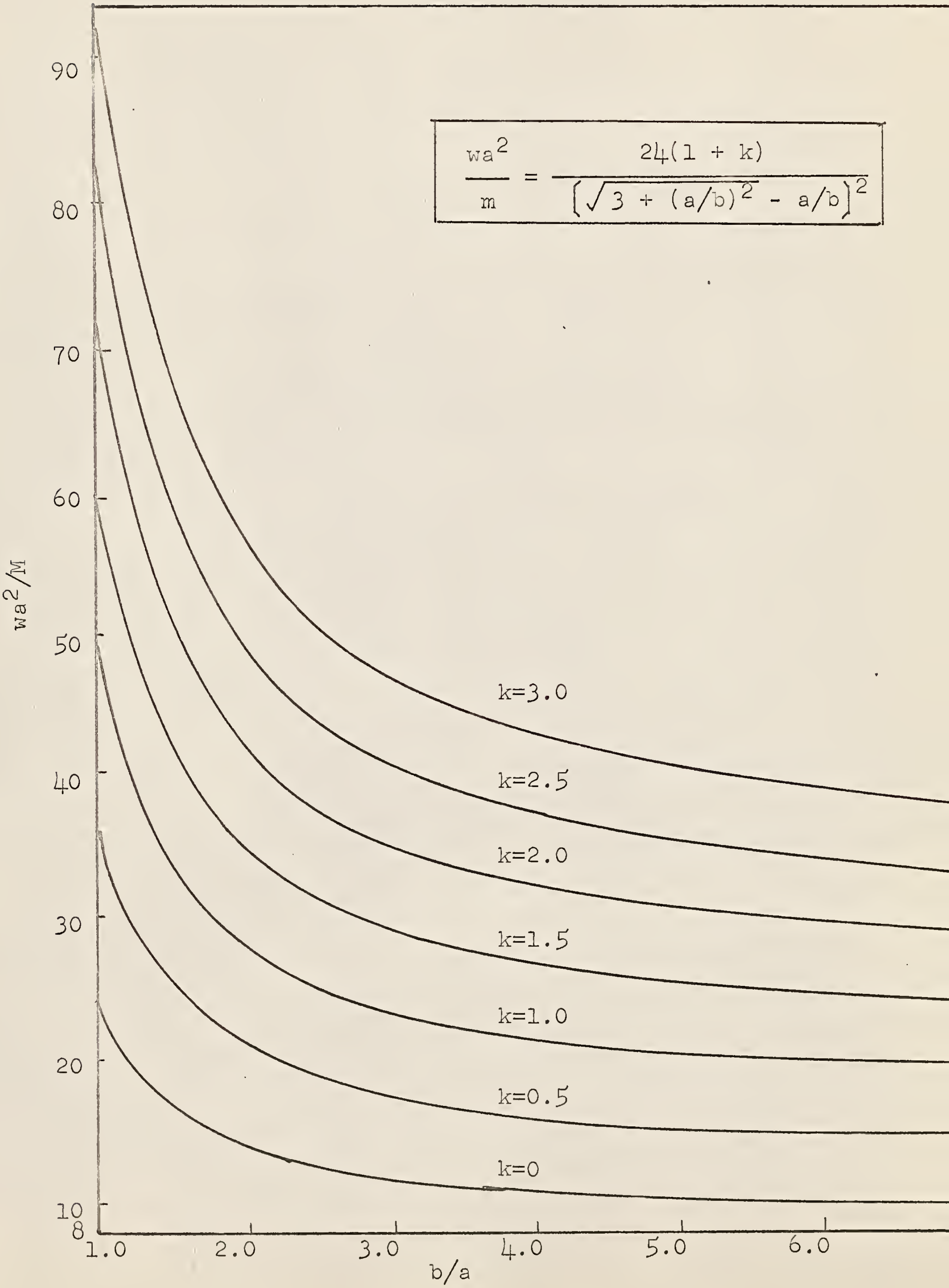
	b/a										
	1.0	1.2	1.4	1.6	1.8	2.0	2.5	3.0	5.0	10.0	
k=0	24.0	20.3	17.9	16.2	15.0	14.1	12.6	11.7	10.1	9.0	8.0
k=0.5	36.0	30.4	26.8	24.4	22.6	21.2	18.9	17.6	15.1	13.5	12.0
k=1.0	48.0	40.5	35.7	32.5	30.1	28.3	25.3	23.4	20.2	18.0	16.0
k=1.5	60.0	50.7	44.7	40.6	37.6	35.3	31.5	29.3	25.1	22.5	20.0
k=2.0	72.0	60.8	53.6	48.7	45.1	42.4	37.9	35.1	30.3	27.0	24.0
k=2.5	84.0	71.0	62.6	56.8	52.6	49.4	44.1	41.0	35.2	31.5	28.0
k=3.0	96.0	81.1	71.5	64.9	60.1	56.6	50.5	46.8	40.4	36.0	32.0

$$\frac{wa^2}{m} = \frac{24(1 + k)}{\left[\sqrt{3 + (a/b)^2} - a/b\right]^2}$$

<u>Example</u>: If a rectangular slab that has values of a = 18 feet, b = 32 feet is fixed at all four edges with negative

EXPLANATION OF PLATE I

Values of wa^2/m for rectangular slabs.



$$\frac{wa^2}{m} = \frac{24(1 + k)}{\left[\sqrt{3 + (a/b)^2} - a/b\right]^2}$$
k=3.0
k=2.5
k=2.0
k=1.5
k=1.0
k=0.5
k=0
90
80
70
60
50
40
30
20
10
8
wa²/M
1.0
2.0
3.0
4.0
5.0
6.0
b/a

reinforcement equal to the positive reinforcement, and is subject to a uniformly distributed load of 4 k/ft^2, find the moment capacity required of the slab.

Solution. In the given case,

$$b/a = 32/18 = 1.78$$

$$k = 1.0$$

From Plate I,

$$wa^2/m = 15.3$$

or

$$m_p = m = wa^2/15.3$$
$$= (18^2 \cdot 4)/15.3$$
$$= 88.5 \text{ k-ft.}$$

VIRTUAL WORK METHOD

The solution of equations of equilibrium may at times be simplified by application of the principle of virtual work. If the yield-line pattern is assumed known, by introducing parameters, such as u and v in Fig. 4, the position of the axes of rotation as well as the ratios between the angles of rotation for the various slab parts are known. Virtual displacement δ may be chosen so that rotations take place only in the yield lines. Virtual work of the pairs of concentrated shears m_t is then zero for the slab as a whole. Virtual work of the yield moments for each individual slab part is the scalar product of two vectors, a rotation $\vec{\theta}$ and a resultant $\vec{M}$ of the moments in the yield lines. For the slab as a whole, this work of the internal force has to equal the work done due to external loads.

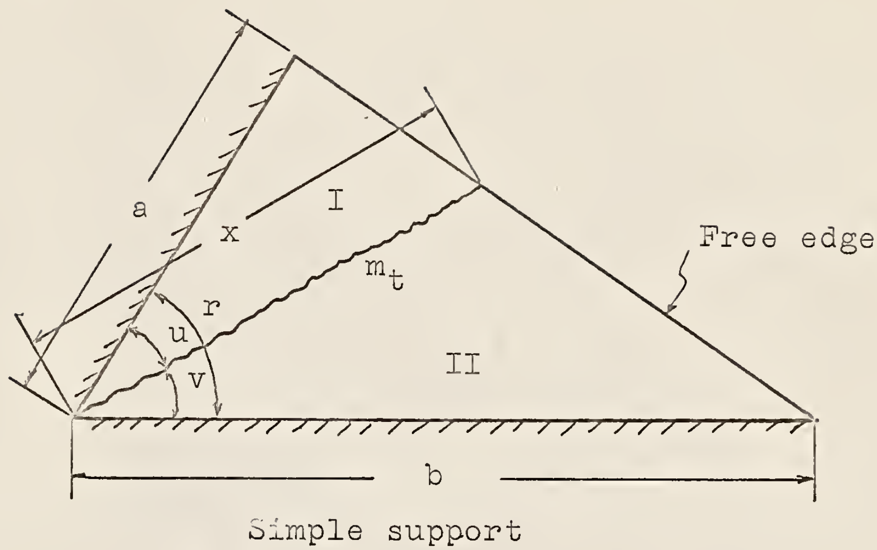


Fig. 4. Triangular slab.

In this manner the principle gives

$$\Sigma \vec{M}\, \vec{\theta} = \Sigma \iint w\delta \; dx \; dy \tag{15}$$

which summation is made over the entire slab, and integration is made over the individual slab parts.

Since $\vec{M}$ is proportional to the unit yield moment m, equation (15) can be used to determine m for a given load w, if the yield-line pattern is known. However, equation (15) can also be used to determine the correct yield-line pattern. The moment across the yield lines being a maximum value, the correct yield pattern corresponding to a load w will give a maximum value of m from equation (15) as compared to other patterns. If a type of pattern is assumed in accord with the support conditions and characterized by a number of unknown parameters x_1, x_2, x_3, . . ., equation (15) may be written

$$m = F(x_1, x_2, x_3, \ldots, w) \tag{16}$$

The correct yield pattern then is formed by the maximum criteria

$$\frac{\partial F}{\partial x_1} = 0 \tag{17a}$$

$$\frac{\partial F}{\partial x_2} = 0 \tag{17b}$$

$$\frac{\partial F}{\partial x_3} = 0 \tag{17c}$$

$$\vdots$$

and the final yield moment m is determined by substituting the corresponding parameter values into equation (16).

Example 1. Find the collapse loads for a rectangular slab (as shown in Fig. 5) with uniform reinforcement, simple supports and subjected to a uniformly distributed load w. With uniform reinforcement the mode of collapse must be symmetrical and in any case the central fracture line must be parallel to the edges since the corresponding axes of rotation never meet (5). The one unknown in this figure is the angle ϕ.

Consider parts A first. The rectangular slab has simple supports and the vector equivalent of two end fracture lines adds up to the short length a. If the maximum deflection is unity, the axes of reference x and y are parallel to the edges, the θ_y, the rotation about axis y, is equal to $\dfrac{1}{a/2 \tan \phi}$ $= \dfrac{2}{a \tan \phi}$ and $\theta_x = 0$.

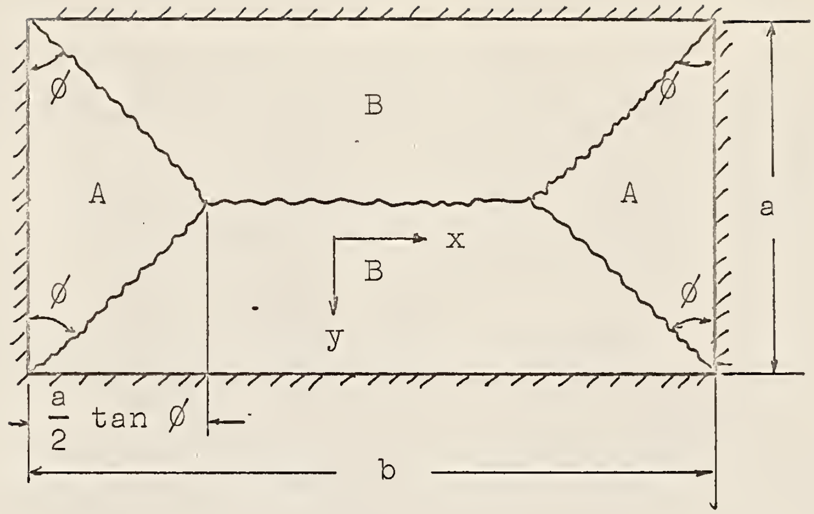


Fig. 5. The rectangular slab.

For parts B of the slab, $\theta_x = 2/a$, and $\theta_y = 0$.

For the whole slab we may express the contributions to the dissipation of internal work W_1 vectorially as follows:

$$W_I = M(\int \vec{\theta}_y \, \vec{dy} + \int \vec{\theta}_x \, \vec{dx})$$

$$= 2(M \cdot a \cdot \frac{2}{a \tan \emptyset} + M \cdot b \cdot \frac{2}{a})$$

$$= 4M(\frac{1}{\tan \emptyset} + \frac{b}{a}) \qquad (18)$$

The external work,

W_E = w (volume of two end half pyramids + central portion)

$$= \frac{1}{3} wa^2 \tan \emptyset + \frac{1}{2} wa(b - a \tan \emptyset)$$

$$= \frac{1}{2} wa^2 \left(\frac{b}{a} - \frac{\tan \phi}{3} \right) \tag{19}$$

Evidently

$$M = \frac{wa^2}{8} \left[\frac{\frac{b}{a} - \frac{\tan \phi}{3}}{\frac{1}{\tan \phi} + \frac{b}{a}} \right] \tag{20}$$

Let $\tan \phi = t \qquad b/a = \lambda$

$$M = \frac{wa^2}{8} \left[\frac{\lambda - t/3}{1/t + \lambda} \right] \tag{20a}$$

There is only one value of t for which w is minimum, or M the required plastic moment is a maximum.

Putting

$$\frac{dM}{dt} = 0$$

leads to
$$- \frac{1}{3} \left(\frac{1}{t} + \lambda \right) - \left(\lambda - \frac{t}{3} \right) \left(- \frac{1}{t^2} \right) = 0 \tag{21}$$

or
$$t^2 = 3 \cdot \frac{(\lambda - t/3)}{(1/t + \lambda)}$$

which we note from equation (20a) is exactly equal to $\frac{24 M}{wa^2}$.

Therefore

$$M = \frac{wa^2}{24} t^2 = \frac{wa^2}{24} \tan^2 \phi \tag{22}$$

Equation (21) is a quadratic in $\tan \phi$ and leads to

$$\tan \phi = \sqrt{3 + \frac{1}{\lambda^2}} - \frac{1}{\lambda} \tag{23}$$

Therefore

$$M = \frac{wa^2}{24}\left[\sqrt{3 + (a/b)^2} - a/b\right]^2 \qquad (24)$$

Equation (24) is identical to equation (9).

Example 2. Consider a regular n-sided slab, each simple support being of length a forming a symmetrical collapse mechanism with an angle ϕ in the plane between each yield line. (See Fig. 6.) Then if the center deflects a unit distance the angle of rotation about each support will be $1/r$, where r is the inscribed radius, and the vector length for the yield lines bounding each triangular portion is $a = 2\,r \tan\frac{\phi}{2}$.

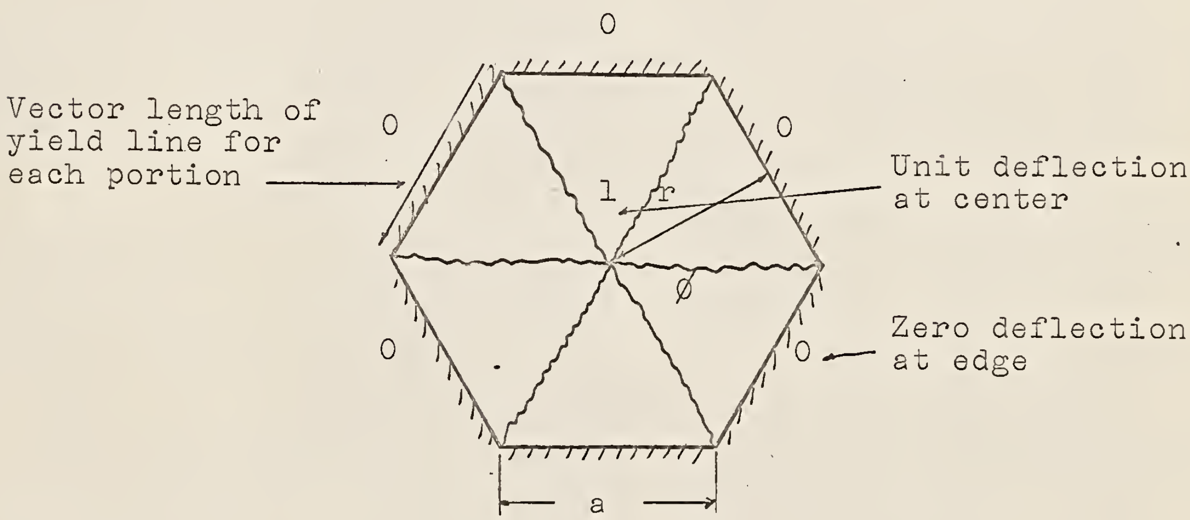


Fig. 6. Regular n-sided (n = 6) slab, each side of length a, on simple supports.

Clearly the dissipation of energy is

$$W_I = n\,2\,r \tan\frac{\phi}{2} \cdot \frac{1}{r} M = 2\,Mn \tan\frac{\pi}{n} \qquad (25)$$

since $\phi = \frac{2\pi}{n}$.

Likewise, the external work W_E for a uniformly distributed load is given by

$$W_E = w \text{ (volume of deflection figure)}$$

$$= \frac{1}{3} w \cdot n \frac{ar}{2} \text{ (unit deflection)}$$

$$= \frac{1}{3} w \cdot n\, r^2 \tan \frac{\pi}{n} \qquad (26)$$

Hence

$$M = \frac{1}{6} w\, r^2 \qquad (27)$$

Thus for the equilateral triangle this leads to $M = \frac{wa^2}{72}$, and for a square $M = \frac{wa^2}{24}$ as before. If n tends to infinity equation (27) clearly expresses the solution for a conical collapse mode of a circular simply supported slab.

If there were a point load P at the center, the external work is simply P x 1 so that

$$P = 2\, Mn \tan \frac{\pi}{n}$$

and $\frac{\pi}{n}$ becomes a very small angle if n is large. In the limit when $n \rightarrow \infty$, expanding $\tan \frac{\pi}{n}$ yields

$$P = 2\, Mn\left(\frac{\pi}{n} + \frac{\pi^3}{n^3} + \ldots\right)$$

or

$$P = 2\pi M = 6.28\, M$$

for a conical collapse mode with a point load acting alone when there is no negative (top) reinforcement.

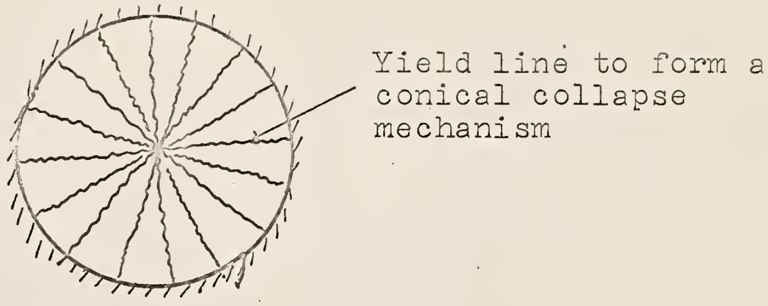


Fig. 7. Circular fan at yield line under a point load.

Example 3. Consider a square slab with three edges simply supported and the fourth free (Fig. 8). The yield lines are now attracted toward the free edge and with this mode the only unknown is the distance y. The relevant rotations of the individual parts are shown in the diagram.

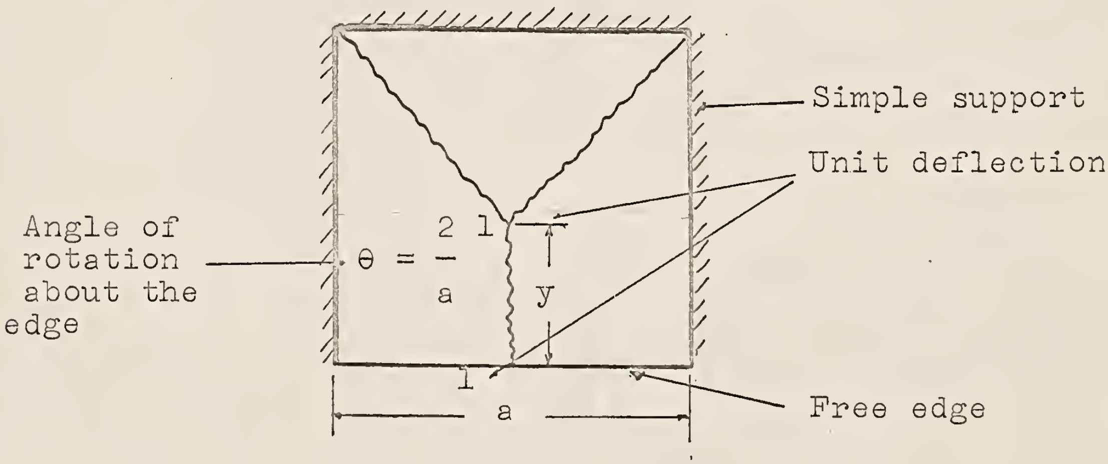


Fig. 8. Collapse of square slab with a free edge.

By the vector procedure it is an easy matter to write the work equation as follows:

$$2\left[Ma\,\frac{2}{a}\right] + Ma \cdot \frac{1}{a - y} = \left[\frac{a}{3}\,(a-y) + \frac{a}{2}\,y\right] \cdot w$$

Putting

$$y/a = Y$$

we obtain

$$\frac{wa^2}{6\,M} = \frac{4 + \dfrac{1}{1 - Y}}{2 + Y}$$

For minimum load

$$\frac{d\left(\dfrac{wa^2}{6\,M}\right)}{dY} = 0$$

$$\frac{\dfrac{2 + Y}{(1 - Y)^2} - \left(4 + \dfrac{1}{1 - Y}\right)}{(2 + Y)^2} = 0$$

or

$$\frac{(2 + Y)}{(1 - Y)^2} - \left(4 + \frac{1}{1 - Y}\right) = 0$$

leading to

$$Y = \frac{5}{4} \pm \quad \frac{25}{16} - \frac{3}{4}$$

Since Y must be less than unity we take the negative sign, and find

$$Y = 0.347$$

whence

$$wa^2 = 14.15\ M$$

It is necessary, however, to check whether the alternative mode of Fig. 9 could take place. Proceeding as before,

$$M\left(2a \frac{1}{x} + 2x \frac{1}{a}\right) = w \left[2 \cdot \frac{1}{3} \cdot ax + \frac{1}{2} a(a - 2x)\right]$$

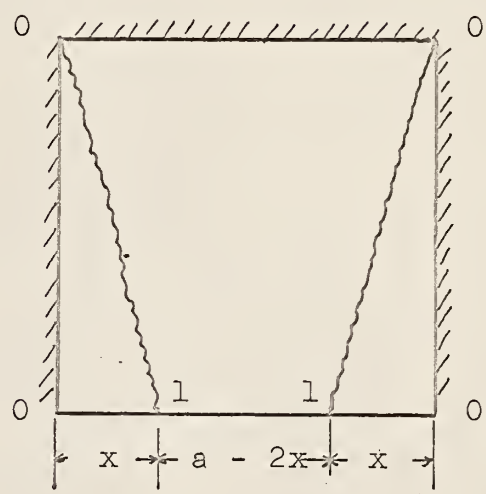


Fig. 9. Collapse of square slab with a free edge.

It is interesting to note here that the combined vector lengths for the two yield lines of the largest portion of the slab just come to 2x--there is a gap in the middle where no work is done. Putting

$$X = x/a$$

we find

$$\frac{wa^2}{2\ M} = \frac{X + \frac{1}{X}}{\frac{1}{2} - \frac{X}{3}}$$

$$\frac{d\left(\dfrac{wa^2}{2\ M}\right)}{a\ X} = 0$$

$$\frac{\left(\dfrac{1}{2} - \dfrac{X}{3}\right)\left(1 - \dfrac{1}{X^2}\right) + \left(X + \dfrac{1}{X}\right)\dfrac{1}{3}}{\left(\dfrac{1}{2} - \dfrac{X}{3}\right)^2} = 0$$

or

$$\left(\frac{1}{2} - \frac{X}{3}\right)\left(1 - \frac{1}{X^2}\right) + \left(X + \frac{1}{X}\right)\frac{1}{3} = 0$$

Simplified

$$X^2 + \frac{4}{3} X - 1 = 0$$

$$X = \frac{-\dfrac{4}{3} + \sqrt{\dfrac{16}{9} + 4}}{2} = 0.533$$

which is an impossible result since X cannot exceed 0.5. So Fig. 8 will be the only correct mode, and $w \cdot a^2 = 14.15$ m will be the correct solution.

Example 4. Consider a triangular slab with one edge free (Fig. 10). Take as axes of reference the two edges for parts A and B, and suppose there is unit deflection at the point where the yield line meets the edge. The orientation of the yield line is as yet unknown. Let the length of the yield line be x. Then the rotation of part A about edge CD

$$\theta_A = \frac{1}{x \sin u}$$

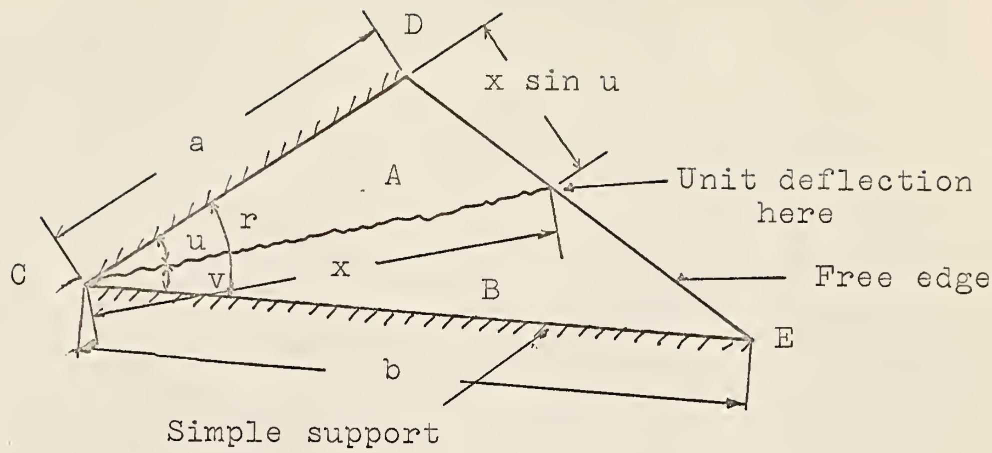


Fig. 10. Simple failure of triangular slab with a free edge.

and part B about edge CE

$$\theta_B = \frac{1}{x \sin v}$$

Hence

$$W_I = M \cdot x \cdot \cos u \cdot \frac{1}{x \sin u} + M \cdot x \cdot \cos v \cdot \frac{1}{x \sin v}$$

$$= M(\cot u + \cot v)$$

$$W_E = w(\frac{\text{area of triangle}}{3})(\text{unit deflection})$$

$$= w \cdot \frac{ab \sin r}{6}$$

Now

$$M(\cot u + \cot v) = M(\frac{\cos u \sin v + \cos v \sin u}{\sin u \sin v})$$

$$= \frac{M \sin(u + v)}{\sin u \sin v}$$

$$= \frac{M \sin r}{\sin u \sin v}$$

Whence

$$M = w \frac{ab}{6} \sin u \sin v$$

$$= \frac{wab}{6} \sin u \sin(r - u)$$

$$\frac{dM}{du} = \frac{wab}{6} \left[- \sin u \cos(r-u) + \sin(r-u) \cos u \right] = 0$$

for a maximum M from which $u = \frac{r}{2}$, i.e., the fracture line always bisects the angle of the triangle. Therefore

$$M = w \frac{ab}{6} \sin^2 \frac{r}{2} = \frac{1}{6} \text{(total load)} \tan \frac{r}{2}$$

since the total load $= \frac{w}{2} ab \sin r$

$$= wab \sin \frac{r}{2} \cdot \cos \frac{r}{2}$$

There is no restriction on the shape of the triangle so that the fracture line and the edge might intersect at any angle.

SUCCESSIVE-APPROXIMATION METHOD

Though the analysis of slabs on an ultimate strength basis is reduced to algebra and geometry as compared to the complex differential equations often resulting from the use of theory of elasticity, solution of the equilibrium equations may be quite time-consuming. The yield-line theory is an extremely powerful analytical tool, and analysis of even complex slabs becomes possible. A practical design method is therefore desirable, which in addition to eliminating the use of differential equations will reduce design work to simple algebra.

The problem in design is generally to estimate the necessary yield moment m for a slab subject to given ultimate loads and with given dimensions and support conditions. The correct value of m is a maximum value resulting from the correct yield pattern and satisfying the equations of equilibrium. It may be shown, therefore, that application of the virtual work principle to yield patterns which do not differ considerably from the correct pattern will result in yield moments only slightly smaller than the correct one (4).

Accordingly, a yield pattern is assumed which is in accord with the support work equation, equation (15). Since for the correct yield pattern all m-values should be equal, a check on the originally assumed yield pattern may be obtained by computing m for every individual slab part from equilibrium equations. If the m-values thus computed differ considerably, the separate values will indicate how the pattern should be altered, and the

first estimate of m from equation (15) will indicate how much the pattern should be changed. With some experience a designer will generally assume a yield pattern the first or second time, which gives a yield moment only a few per cent in error.

This practical method may be illustrated by an analysis of the following problem.

The floor slab (in Fig. 11) is fixed or continuous on three edges, a negative reinforcement equal to the positive reinforcement being chosen, which gives m' = m. The fourth edge is simply supported, m' = 0, and an opening with a a free edge is provided for a staircase. The loads indicated represent service loads multiplied by proper load factors. Thus a uniform load w = 200 psf, a line load of 250 pounds per foot from a partition wall, and a line load of 1,000 pounds per foot from the staircase

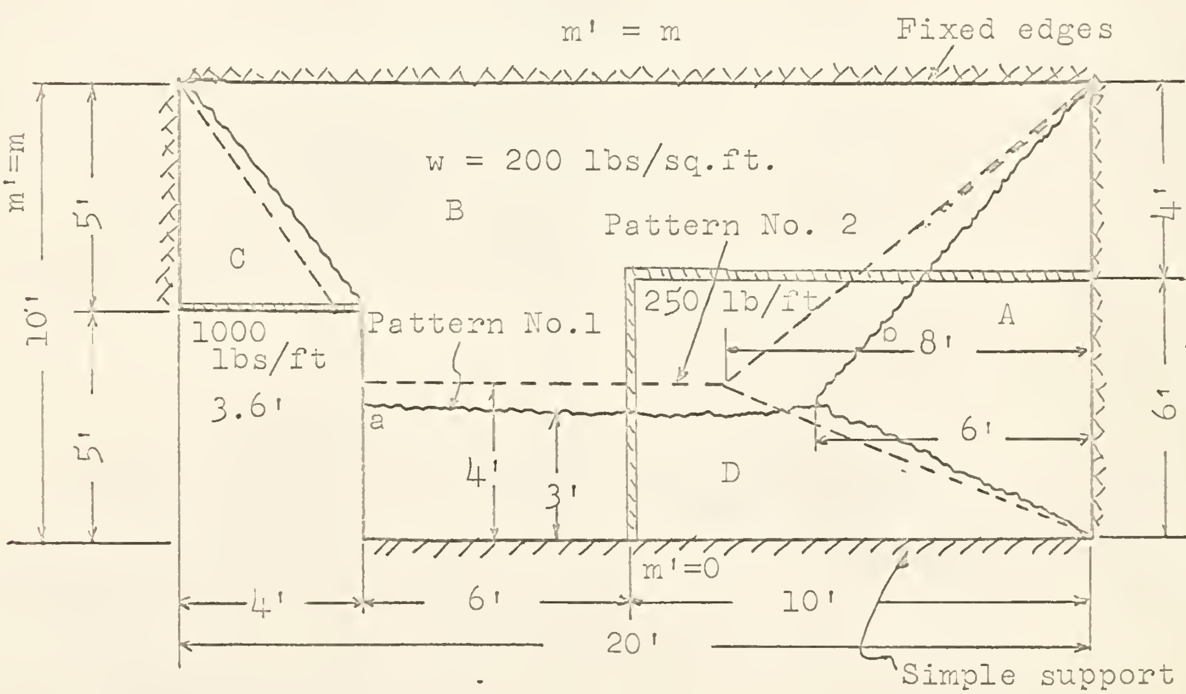


Fig. 11. Sketch of floor slab.

are given. An elastic analysis of such a slab would be extremely difficult even with the aid of approximate numerical procedures.

A yield pattern is assumed, indicated as pattern No. 1 in Fig. 11. The assumption is guided by the fact that yield lines between two slabs must pass through the intersection of the corresponding axes of rotation, that is, the supports. There is no moment at the simply supported or the free edges. Simply supported or free edges, as well as openings, attract yield lines while fixed edges repel them.

The necessary yield moment m is first computed for each slab part separately by equilibrium of moments about the supports.

Part A.

$$10(m+m) = 200 \times 10 \times \frac{6^2}{6} + 250 \times \frac{3.43^2}{2}.$$

$$20\ m = 13{,}470 \quad \text{or} \quad m_A = 674 \text{ lb-ft.}$$

Part B.

$$20(m+m) = 200(6 \times \frac{7^2}{6} + 10 \times \frac{7^2}{2} + 4 \times \frac{5^2}{6})$$

$$+ 250(6.57 \times 4 + 3 \times 5.5) - \frac{4}{5} m \times 5$$

$$40(m) = 72.900 - 4m$$

$$m_B = 1653 \text{ lb-ft.}$$

Part C.

$$5(m+m) = 200 \times 5 \times \frac{4^2}{6} + 1000 \times \frac{4^2}{2} + \frac{4}{5} m \times 4$$

$$10\ m = 10.670 + 3.2\ m$$

or $$m_C = 1568 \text{ lb-ft.}$$

Part D.

$$16(m+0) = 200(6 \times \frac{3^2}{6} + 10 \frac{3^2}{2}) + 250 \times \frac{3^2}{2}$$

$$16\ m = 12.030 \quad \text{or } m_D = 753 \text{ lb-ft.}$$

It is seen that the m-values vary from 674 to 1653 pounds and the assumed yield pattern is therefore not the correct one. An estimate of the correct yield moment may nevertheless be obtained by applying the virtual work equation. A virtual deflection of unity along the yield line a-b gives the rotations:

$$\theta_A = 1/6$$

$$\theta_B = 1/7$$

$$\theta_C = \frac{5/7}{4} = \frac{1}{5.6}$$

$$\theta_D = 1/3$$

The equilibrium equations above are written in such a manner that the virtual work equation can be established easily.

$$W_I = \sum \vec{M}\ \vec{\theta} = m(\frac{20}{6} + \frac{40}{7} + \frac{10}{5.6} + \frac{16}{3}) = 16.17\ m$$

$$W_E = \sum \iint w\delta\ dx\ dy = \frac{13.470}{6} + \frac{72.490}{6} + \frac{10.670}{5.6} + \frac{12.030}{3}$$

$$= 18.560$$

$$W_I = W_E$$

which gives

$$m_1 = \frac{18.560}{16.17} = 1148 \text{ lb-ft.}$$

for yield pattern No. 1.

It is found that m_A and m_D are less than m_1; and m_B and m_C

are greater than m_1; therefore the area of slab parts A and D must be increased and that of slab parts C and D must be decreased. Such correction leads to pattern No. 2, in Fig. 11, which gives:

$$M_A = 1,243 \text{ lb-ft.}$$

$$M_B = 1,182 \text{ lb-ft.}$$

$$M_C = 1,167 \text{ lb-ft.}$$

$$M_D = 1,190 \text{ lb-ft.}$$

The corresponding virtual work equation gives

$$m_2 = 1,192 \text{ lb-ft.}$$

In this case the four m-values are almost equal and m = 1,192 lb-ft. is a satisfactory design value. It should be noticed that m_2 is only 3.8 per cent greater than m_1. Application of the virtual work equation to yield patterns reasonably close to the correct one gives yield moments only slightly less than the correct value.

ORTHOGONALLY ANISOTROPIC REINFORCEMENT AND AN ECONOMIC STUDY OF REINFORCEMENT

A slab which has resisting moments in two perpendicular directions that are not equal is called orthogonally anisotropic. The analysis of two-way slabs presented herein has so far been concerned only with slabs having an equal yield moment in all directions. Such isotropic reinforcement is often not economical, and methods of analysis for anisotropic reinforcement are therefore desirable. Johansen (4) has shown that the analysis of slabs with yield moments m and $m' = k \cdot m$ in one direction and μm, $\mu m' = k\mu m$ in an orthogonal direction may be reduced to the case of isotropic reinforcement $\mu = 1.0$.

A slab part with positive and negative yield lines is shown in Fig. 12, the co-ordinate axes being in the directions of m and μm. It is assumed that the negative moments m' and $\mu m'$ have the same directions. The resultant of the positive moments acting on the slab part shown is the vector $\vec{a}$ times the yield moment in its direction, and the resultant moment components in the x and y directions are $M_x = m \cdot a_x$ and $M_y = \mu m \cdot a_y$ respectively, in which a_x and a_y are components of the vector $\vec{a}$. In the same manner the negative moments give $M_x' = m'b_x$ and $M_y' = \mu m' b_y$.

Let the axis of rotation R-R of this rigid portion R intersect the x-axis at an angle α, such that if the rotation of the rigid portion about R-R is θ_R, the component rotations are θ_x and θ_y. Line R-R is a line of constant deflection, and will pass through a column support. There is a distributed load of

intensity w, a line load of intensity $\overline{w}$ and of length ℓ inclined at ψ to the x-axis and a concentrated load of P.

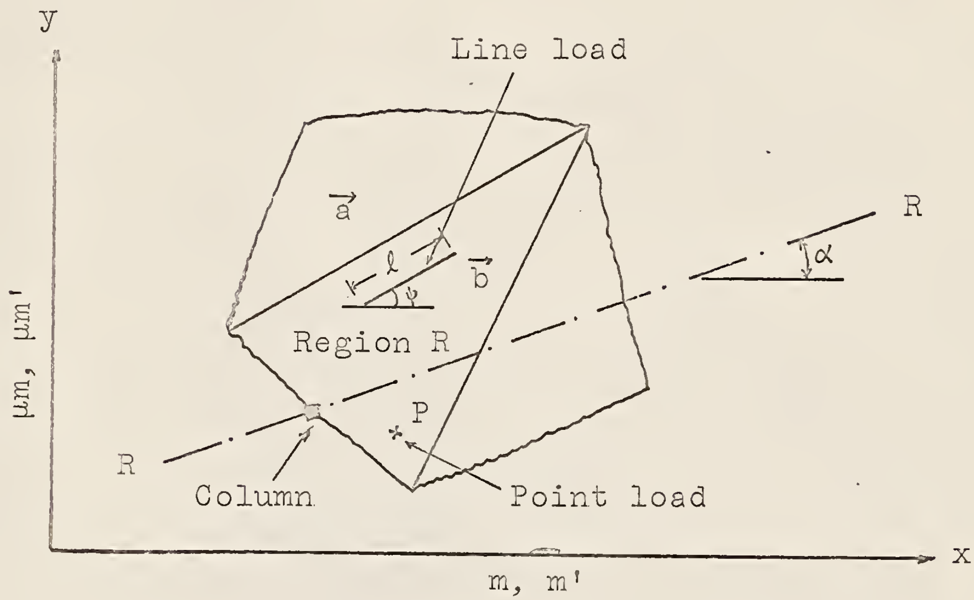


Fig. 12. Slab with orthogonally anisotropic reinforcement.

Then the equation of energy equilibrium becomes

$$(ma_x + m'b_x)\theta_x + (\mu ma_y + \mu m'b_y)\theta_y$$
$$= \iint zwdx\ dy + \int_0^{\ell_x} \overline{w}\ \overline{z}\ ds + Pz_p \qquad (28)$$

where z is the deflection at a point x, y; $\overline{z}$ is the deflection at a point on the line load; and z_p is the deflection of the point load.

Considering the line load in more detail and observing that $ds^2 = dx^2 + dy^2$, then

$$\int_0^{\ell} \overline{w}\ \overline{z}\ ds = \int_0^{\ell x} \overline{w}\ \overline{z} \sqrt{1 + \frac{dy^2}{dx^2}} \cdot dx$$

$$= \int_0^{l_x} \bar{w}\,\bar{z}\,\sqrt{1 + \tan^2 \phi}\; dx \qquad (29)$$

Now consider a slab with positive plastic moment m in both directions and likewise negative moments m' in both directions, and suppose that this slab has all its dimensions in the y-direction multiplied by k. Then in order to have corresponding deflections at corresponding points, we make the rotation θ_R' of the region R in the slab have the components $\theta_x' = \theta_x/k$; $\theta_y' = \theta_y$.

Hence the equation of energy equilibrium is

$$(mka_y + m'kb_y)\theta_y + (ma_x + m'b_x)\theta_x/k$$

$$= \iint w'z\; dx\; k\; dy + \bar{w}'\bar{z}\; ds' + P'z_p \qquad (30)$$

where w', $\bar{w}'$, P' are corresponding loads on the slab.

Then dividing both sides by k we get typical expressions,

$$(ma_y + m'b_y)\theta_y + (ma_x + m'b_x)\,\frac{\theta_x}{k^2}$$

$$= \iint w'z\; dxdy + \bar{w}'\bar{z}\,\frac{ds'}{k} + \frac{P'z_p}{k} \qquad (31)$$

It will now be observed that these equations are exactly the same as for the original (orthotropic) slab provided we make the following transformations. This is the so-called "affinity theorem" that was developed by Hognestad (9) and Johansen (10).

(1) Length ratio $k^2 = 1/\mu$, or $k = 1/\ \mu$.

(2) Distributed load, $w' = w$, i.e., the same intensity of load.

(3) Concentrated load, $P'\sqrt{\mu} = P$, or $P' = P/\sqrt{\mu}$.

(4) Line load; observing that

$$ds' = \sqrt{dx^2 + k^2(dy)^2}$$

then

$$\frac{ds'}{R} = dx\sqrt{\frac{1}{k^2} + \tan^2\psi}$$

Hence we must make

$$\bar{w}' = \sqrt{\frac{1 + \tan^2\psi}{\mu + \tan^2\psi}} \cdot \bar{w}$$

which simplifies to

$$\bar{w}' = \frac{\bar{w}}{\sqrt{\mu \cos^2\psi + \sin^2\psi}}$$

Example 1. Consider a triangular slab (Fig. 13) with 30 degrees, 60 degrees, 90 degrees, and a free edge opposite the right-angled corner, the other sides being simply supported. Lengths of sides 10 feet, 5.77 feet, 11.54 feet. There is a line load of intensity $\bar{w}$ along the free edge. Find the collapse load, w , ignoring corner effects, if there is a plastic moment of m for the reinforcement parallel to the 10-foot side and 4m[1] at right angle to it.

---

[1]In orthotropic reinforcement problems, μ can be assumed to be any value theoretically. The assumption for the value of μ does not assure the most economical section. Values of μ which result in the most economical sections are discussed on pages 39-41 . Here it is assumed $\mu = 4$ only for convenience, because $\sqrt{\mu} = \sqrt{4} = 2$.

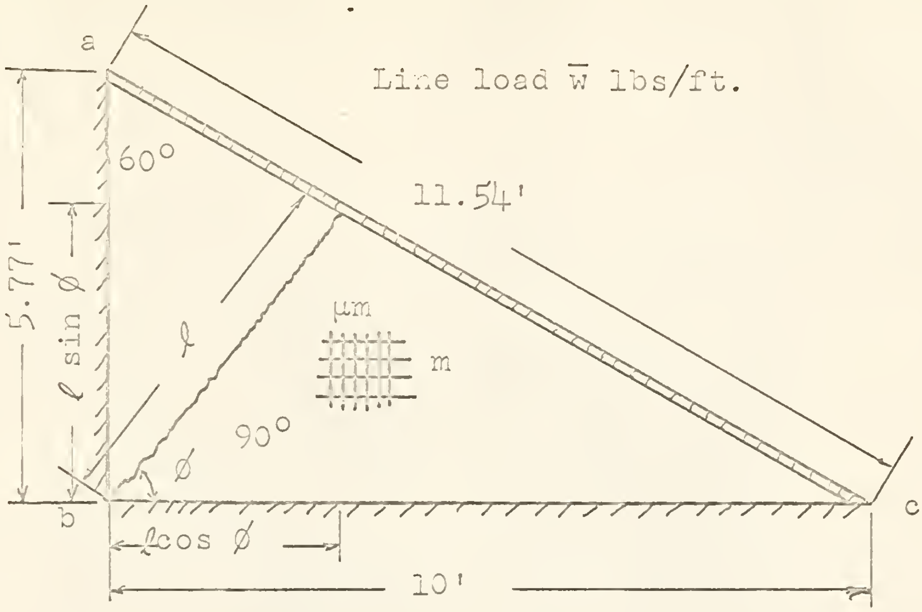


Fig. 13a. Original orthotropic triangular slab.

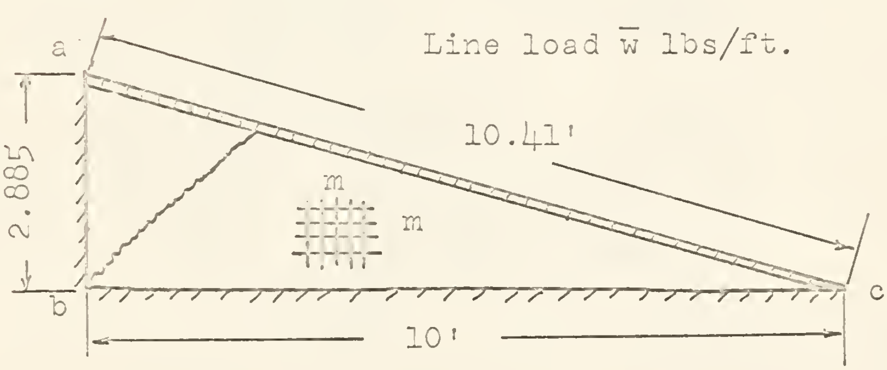


Fig. 13b. Equivalent isotropic triangular slab.

(i) Solving by virtual work (see Fig. 13a). Taking the deflection where the yield line meets the edge to be unity, then

$$W_I = 4\, m \ell \cos \phi \cdot \frac{1}{\ell \sin \phi} + m \cdot \ell \sin \phi \frac{1}{\ell \cos \phi}$$

$$= 4\, m \cot \phi + m \tan \phi$$

whatever the separate lengths of line loads are, the centers of gravity deflect a distance of 1/2 so that

$$W_E = 11.54 \cdot \overline{w} \cdot 1/2 = 5.77\, \overline{w}$$

$$W_I = W_E$$

Hence

$$W/m = 0.693 \cot \phi + 0.173 \tan \phi \qquad (32)$$

$$\frac{d(w/m)}{d\phi} = -\ .693 \csc^2\phi + 0.173 \sec^2\phi = 0$$

$$\tan^2\phi = \frac{0.693}{0.173} = 4$$

leading to $\tan \phi = 2$, or $\phi = 63°30'$ for the inclination of the yield line.

Substituting into equation (32)

$$\overline{w} = 0.693 \cdot \frac{1}{2}\, m + 0.173 \cdot 2\, m$$

$$= 0.693\, m$$

(ii) Solving by the affinity theorem (see Fig. 13b). Consider the slab ab $= 5.77/\sqrt{4} = 2.885$, bc $= 10$; therefore the free edge is 10.41 feet long. For this slab

$$W_I = m \cot \phi + m \tan \phi$$

from previous solution which is minimum when $\phi = 45$ degrees.

$$W_E = 10.41\, \overline{w} \cdot \frac{1}{2}$$

$W_I = W_E$ leading to

$$2m = \frac{1}{2} \cdot 10.41\ \overline{w}'$$

$$\overline{w}' = 0.384\ m$$

Then apply the transformation rules,

$$\overline{w}' = \frac{\overline{w}}{\sqrt{4 \cdot \cos^2 30^\circ + \sin^2 30^\circ}}$$

$$= \frac{\overline{w}}{\sqrt{3.25}}$$

$$= 0.554\ \overline{w}$$

Then

$$\overline{w} = \frac{0.384}{0.554}\ m = 0.693\ m$$

as before.

Example 2. Consider the slab of Fig. 14 and let the resisting moment in the long direction equal m; and, in the short direction, equal μm, where $\mu = 4$. For the simply supported slab there is only one variable as before, namely, the angle ϕ of the yield lines.

Applying the affinity theorem, the transformed slab remains of length a in the m direction, but the length in the μm direction is divided by $\sqrt{\mu}$. Hence the slab has an effective ratio

$$\frac{b'}{a'} = \frac{b}{a}\ \sqrt{\mu} = \frac{b}{a} \cdot \sqrt{4}$$

Assume $\quad b/a = 2.5$

Then $\quad b'/a' = 2.5 \cdot \sqrt{4} = 5.0$

From our previous results equation (9), or directly from Plate I.

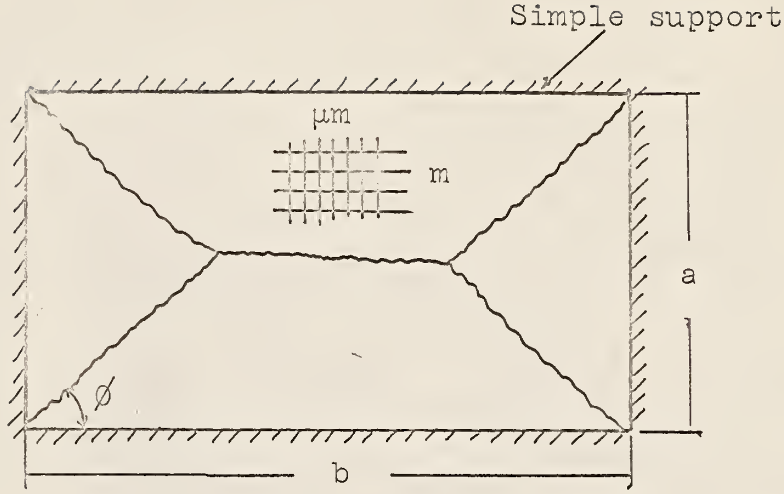


Fig. 14. Simply supported rectangular slab with orthotropic reinforcement.

$$m = \frac{wa^2}{24.0}\left(\sqrt{3 + (\frac{1}{5})^2} - \frac{1}{5}\right)^2$$

$$= \frac{wa^2}{40.4} \qquad \text{in the long direction}$$

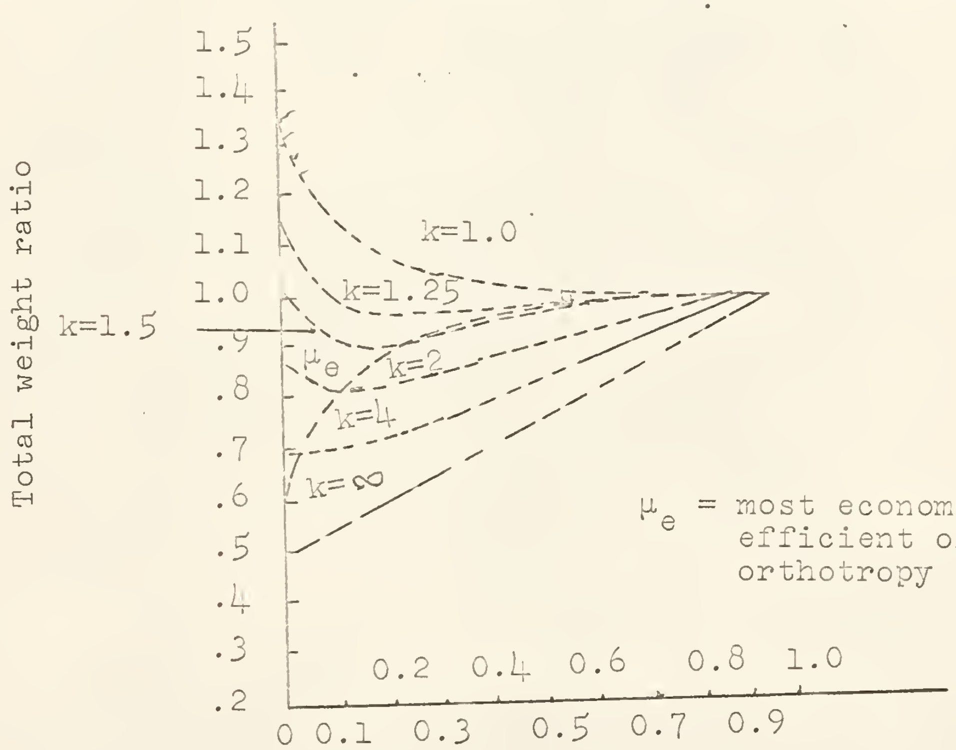
$$\mu m = \frac{wa^2}{10.1} \qquad \text{in the short direction.}$$

From Plate I we also have $m = w\, a^2/12.6$ in both directions with isotropic reinforcement. Comparing these results with each other, we obtain the ratio of total steel

$$r = \frac{\frac{1}{10.1} + \frac{1}{40.1}}{\frac{1}{12.6} \times 2} = 0.78$$

This tells us that the use of orthogonally anisotropic reinforcement will save 22 per cent of the steel.

Now we can evaluate the comparable amounts of steel for all values of $\mu (\mu > 1)$ and for various shapes of rectangles. Wood (5) has found the various results as plotted in Fig. 15, where a total weight ratio of unity means that the same weight of steel would be used for any particular rectangle as would be required for that rectangle if isotropic reinforcement had been used. Values less than unity imply a corresponding saving of steel for the unequally disposed reinforcement. It is convenient to plot the chart on a base of $1/\mu$, since with longitudinal reinforcement omitted altogether the value of μ is ∞. It will be observed that with a 2:1 rectangle some 19 per cent of the



$$\text{Value of } 1/\mu = \frac{\text{longitudinal reinforcement}}{\text{transverse reinforcement}}$$

Fig. 15. Simply supported rectangular slabs: Proportionate weight of reinforcement by using orthotropic reinforcement compared with isotropic reinforcement. Uniform loading.

steel can be saved, but that the most economical coefficient of orthotropy μ is very large, being as much as 10.

We will discuss in more detail the most economical arrangement of reinforcement in two directions for a rectangular slab. Let μ_e be the most economical proportion.

We have noticed in the previous calculation that

$$\mu m = \frac{wa^2}{24} \tan^2 \phi' = \frac{wa^2}{24} t^2$$

where $t = \tan \phi'$. Now the weight of reinforcement per unit area of slab is very approximately proportional to $m + \mu m$. Hence the unit weight

$$W \propto m + \mu m \propto (1 + 1/\mu) t^2$$

Since

$$t = \sqrt{3 + \frac{a^2}{b^2 \mu}} - \frac{a}{b\sqrt{\mu}}$$

we find that

$$\mu = \frac{4t^2}{(3 - t^2)^2} \cdot \frac{a^2}{b^2} \tag{33}$$

and

$$W \propto \frac{(3 - t^2)^2}{4} \frac{b^2}{a^2} + t^2 .$$

Now

$$\frac{dw}{d\mu} = \frac{dw}{dt} \cdot \frac{dt}{d\mu} = 0$$

and observing that $\frac{dt}{d\mu}$ will not be zero within the range of values of μ we are considering, then for minimum weight $\frac{dw}{dt} = 0$.

This obtains

$$3 - t^2 = \frac{2a^2}{b^2} .$$

Substituting into equation (33), we get the most economical value of μ which is

$$\mu_e = 3 \frac{b^2}{a^2} - 2 \qquad (34)$$

This leads to some surprisingly large values of μ_e. For instance, when $b/a = 2$, $\mu_e = 10$; $b/a = 3$, $\mu_e = 25$. Consequently the distribution of reinforcement may be determined by the minimum amounts of reinforcement specified in various codes.

However, the optimum value μ_e can be misleading because it does not indicate just how much superior it is to other alternative values. Figure 15, where a dotted curve representing μ_e is shown, is much better in this respect.

CORNER EFFECT

It has so far been assumed that yield lines enter the corners between two intersecting supported sides without divergence. Tests to failure on slabs indicate that the localized yield-line patterns may occur in the region of the corners and the main yield lines fork as shown in Fig. 16 (6). The small planar elements B rotate about the broken lines and these are referred to as corner levers.

The corner lever appears because the Y-shaped yield-line pattern is more dangerous, that is, gives a higher necessary

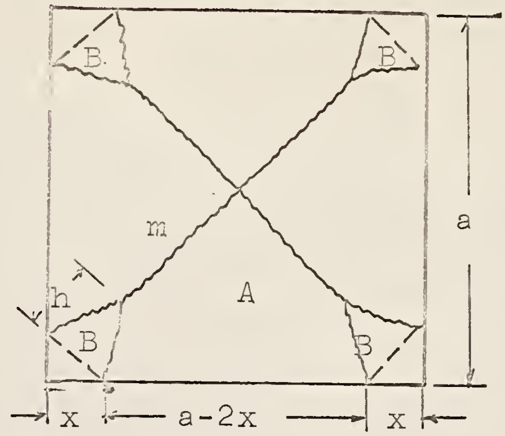


Fig. 16. Square slab with corner levers.

yield moment, than the single yield line. According to Johansen (10) it is most expedient in practical design to disregard the corner levers and then later apply corrections, for which he developed general equations and tabulated the most common cases. In this brief presentation, however, the characteristics of corner levers are probably best illustrated by an example.

Consider a simply supported and uniformly loaded square slab; equation (12) gives $m = wa^2/24$ assuming that no corner levers form. If it is assumed that the corner effects are held down and that symmetrical corner levers exist as indicated in Fig. 16, a yield line with $m' = K\ m$ may be present across the corners. Equilibrium of moments for triangle B about axis b-b then gives

$$x\sqrt{2}\,(m + m') = wx\sqrt{2}\,\frac{h^2}{6}\ ,$$

or

$$h = \sqrt{\frac{m}{w}}\ \sqrt{6(1 + k)}\ . \tag{35}$$

Equilibrium of slab part A about its supported side leads to

$$F(x \cdot m \cdot w) = m(a - 2x) - w\left[\frac{a^3}{24} - \frac{24}{6}\left(\frac{x}{2} + \frac{h}{2}\right)^2\right] = 0 \quad (36)$$

Equation (36) defines m as an implicit function of x and w. For a given w, an x corresponding to a maximum of m is sought. We have the criterion

$$\frac{dm}{ax} = -\frac{\partial F/\partial x}{\partial F/\partial m} = 0$$

or

$$\frac{\partial F}{\partial x} = 0 .$$

Assuming $\partial F/\partial m$ is finite and not equal to zero,

$$\frac{\partial F}{\partial x} = 0 \text{ gives}$$

$$x = \frac{2}{3}\left(\sqrt{\frac{1}{2}h^2 + 18\frac{m}{w}} - \frac{h}{\sqrt{2}}\right) .$$

Substituting h from equation (35),

$$x = \frac{2}{3}\sqrt{\frac{m}{w}}\left[\sqrt{21 + 3k} - \sqrt{12(1 + k)}\right] \qquad (37)$$

The maximum value of m may then be found by solving equations (35), (36), and (37) for m, which solution is algebraically rather cumbersome.

A solution may be obtained easily, however, using successive approximations. An initial estimate of the value m/w is made, $m/w = a^2/24$ being a reasonable value. For this m/w, h and x are computed from equations (35) and (37), and a new value of m/w is found from equation (36). This new m/w is returned to

equations (35) and (37), and so on.

For the various values of $k = m'/m$, the solutions as itemized in Table 2 are found.

Table 2. Values of m/w for square slabs with corner effect.

$k = m'/m$	x	h	m/w
0	0.159 a	0.523 a	$a^2/22.0$
1/4	0.110 a	0.571 a	$a^2/23.0$
1/2	0.069 a	0.619 a	$a^2/23.6$
1	0.000 a	--	$a^2/24.0$

It appears that even a weak corner reinforcement brings the yield moment in square slabs close to the value $m = wa^2/24$ corresponding to $m' = m$, for which the corner levers disappear. When no corner reinforcement is provided, however, the yield moment is increased about 9 per cent. In reality concrete has some flexural strength and the actual influence of the levers will be somewhat smaller than indicated. It should be noted, however, that in some cases the influence of corner levers may be considerably larger than for a square slab. In triangular slabs (Fig. 5) for instance, the absence of corner reinforcement may increase the yield moment 20 to 35 per cent (9).

CONCLUSION

The yield-line theory is intended for the prediction of the ultimate flexural strength of reinforced concrete slabs. The theory seems best suited for ultimate load design. However, experience has indicated that a design based on flexural strength using the yield-line theory must be supplemented by a check of the conditions under service loads, particularly deflections under sustained loading. In common designs this may be achieved through specification of minimum values for reinforcement and for slab thickness.

Although yield-line patterns with corner levers are generally more critical than those without, they are often neglected in yield-line analysis. The analysis becomes considerably more complicated if the possibility of corner levers is introduced, and the error made by neglecting them is usually small. An analysis of the square slab results in a required moment of $w\,a^2/22$, an increase of about nine per cent as compared with the results of an analysis neglecting corner levers.

ACKNOWLEDGMENT

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BIBLIOGRAPHY

1. A History of Civil Engineering, by Hans Straub. The Massachusetts Institute of Technology Press, Cambridge, Massachusetts. 1964. pp. 220-230.

2. Plastic Analysis of Structures, by Philip G. Hodge. McGraw-Hill Book Company, Inc., New York. 1959. pp. 1-56.

3. "Dansk Ingeniorforenings Normer for Bygningskonstruktioner; 2. Beton og jernbetonkonstruktioner," Danish Standard No. 411. Danish Standards Assn., Copenhagen, June, 1949, pp. 34-37.

4. Yield Line Theory, by K. W. Johansen. Cement and Concrete Association, London. 1962.

5. Plastic and Elastic Design of Slabs and Plates, by R. H. Wood. Thames and Hudson, London. 1961. pp. 1-79, and pp. 111-189.

6. Structural Concrete, by J. D. Davies. The Macmillan Company, New York. 1964. pp. 128-149.

7. Plastic Design of Steel Frames, by Lynn S. Beedle. John Wiley & Sons, Inc., New York. 1964. pp. 1-21, and pp. 55-104.

8. Design of Concrete Structures, by George Winter. Seventh edition, Civil Engineering Series, McGraw-Hill, New York. 1964. pp. 231-242.

9. "Yield-line Theory for the Ultimate Flexural Strength of Reinforced Concrete Slabs," by Eivind Hognestad. Journal, American Concrete Institute. March, 1953. pp. 637-654.

10. "Beregning af krydsarmerede Jaernbetonpladers Brudmoment," by K. W. Johansen. Bygningsstatiske Meddelelser (Copenhagen) V. 3, 1931. pp. 1-18.

11. "Building Code Requirements for Reinforced Concrete," ACI Standard 318-63, ACI Committee 318, American Concrete Institute. June, 1963.

APPENDIX

APPENDIX

An Assigned Problem. Design a rectangular slab 24 feet by 16 feet to support a uniform live load of 180 psf. The edge conditions are to be as shown on Fig. 17. A 4,000-psi concrete and steel with fy = 40,000 psi are to be used.

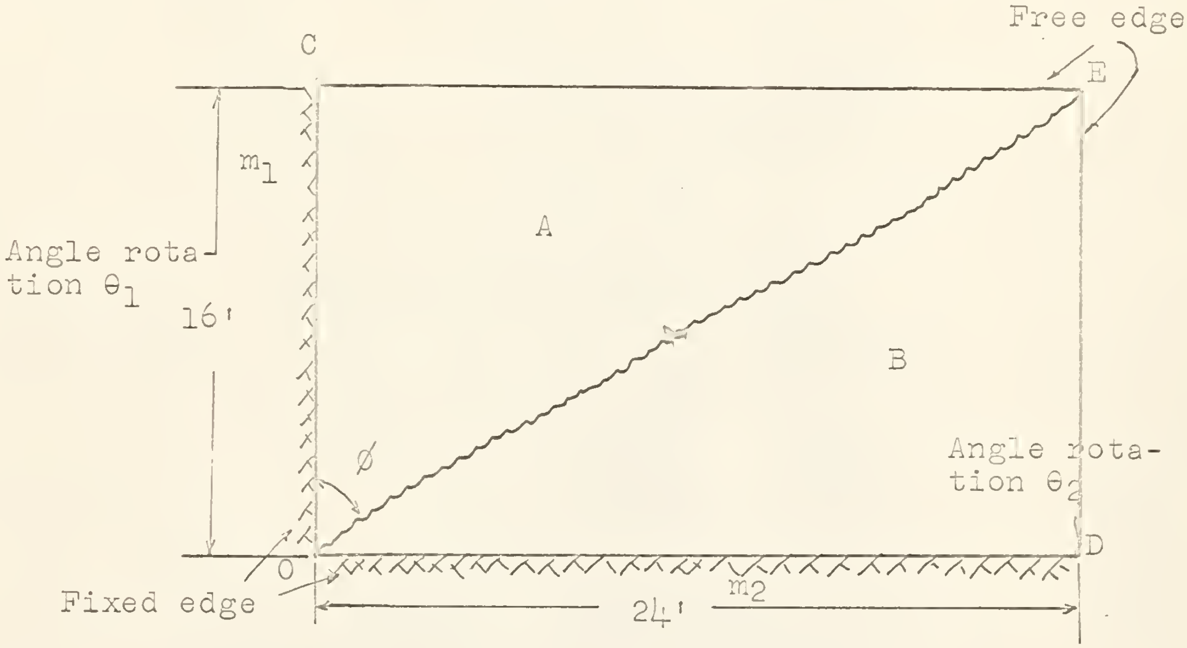

Fig. 17. A rectangular slab with two adjacent free edges.

The minimum thickness, by which ACI Code (11) of $\dfrac{(24 + 16)^2}{180}$ = 0.445 foot. Use 9 inches; because of heavy loads, deeper slabs lead to steel savings which frequently prove more economical than minimum thickness slabs. Consequently the dead load is 9/12 x 150 = 112.5 psf, and the design loads for ultimate strength design are

Live load:	1.8 x 180 =	324 psf
Dead load:	1.5 x 112.5 =	169 psf
Total		493 psf

It is known[1] that to use orthogonal anisotropic reinforcement is more economical than to use the isotropic type. Then observe the rules[2] that: (1) The yield line must pass through the point of intersection of the axes of rotation of the two adjacent slab segments; and (2) the yield line must be straight. We may now assume any straight line that passes through the point of intersection O to be a correct yield line. The angle θ (measured from edge $\overline{OC}$ to the central yield line) may vary from 0 degrees to 90 degrees theoretically, depending only on what ratio of moment capacity in the two orthogonal directions we prefer. In other words, the tangent of angle θ is proportional to the moment capacity ratio in the two directions; the angle θ will increase as the moment capacity ratio of the two directions increases. For convenience, the one that goes along the diagonal of the rectangular slab has been chosen.

Then applying the principle of virtual work, we have the energy equilibrium $W_I = W_E$. Substituting,

$$(16 + 16)m_1\theta_1 = 1/2 \times 16 \times 24 \times 493 \times 1/3 \times 24\ \theta_1$$

or $\quad m_1 = 23{,}7000$ ft-lbs = 284,000 in-lbs.

Likewise

$$(24 + 24)m_2\theta_2 = \frac{1}{2}\ 16 \cdot 14 \cdot 493 \cdot \frac{1}{3} \cdot 16\ \theta_2$$

---

[1]See discussion on page 37.

[2]See page 5.

or $m_2 = 10{,}500$ ft-lbs $= 126{,}000$ in-lbs.

Following the ACI Building Code 318-63 (11),

$$\frac{m_1}{\phi bd^2} = \frac{284{,}000}{0.9 \times 12 \times 7^2} = 537 .$$

By graph on page 634 of reference (7),

$$p = 0.0146$$

and

$$A_s = 0.0146 \times 12 \times 7 = 1.23 \text{ sq.in.}$$

Use No. 6 bars at a spacing of 4 inches c. to c., both top and bottom along the long span direction.

$$\frac{m_2}{\phi bd^2} = \frac{126{,}000}{0.9 \times 12 \times 7.7^2} = 212$$

$$p = 0.0054$$

and

$$A_s = 7.7 \times 12 \times 0.0054 = 0.5 \text{ sq.in.}$$

Use No. 6 bars at a spacing of 6 inches c. to c., both top and bottom along the short span direction.

This seems to be a satisfactory pattern of reinforcement. Designs for other yield lines may be tried and analyses made to compare the relative economics of the resulting slabs.

Details of the rectangular slab which was designed are shown in Fig. 18. The reinforcement in both directions is specified on the plan in the usual manner, and two typical sections are shown. The short direction reinforcement is placed on the outside of the long direction bars as the moment in the short direction is less than that in the long direction.

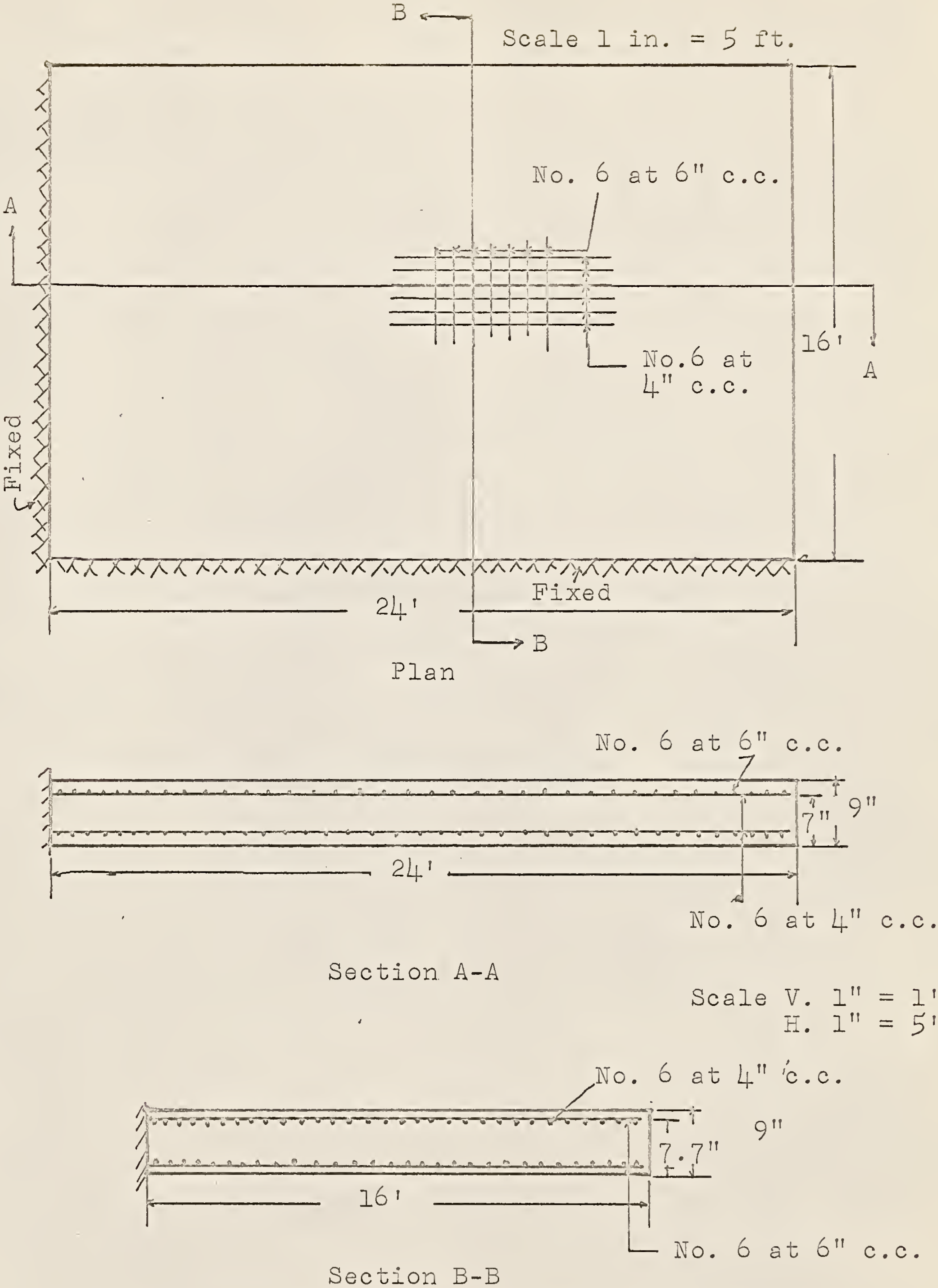

Fig. 18. Details of the rectangular slab design.

THE YIELD-LINE THEORY FOR CONCRETE SLABS

by

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AN ABSTRACT OF A MASTER'S REPORT

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An outline of the yield-line theory, a plastic theory for the prediction of ultimate flexural strength of reinforced concrete slabs, developed by K. W. Johansen, is presented. The theory is based on plastic behavior occurring in a pattern of yield lines, the location of which depends on loading and boundary conditions. The ultimate flexural strength may be evaluated, even for complex slabs, with limited mathematical effort. The theoretical strengths obtained are in good agreement with experimental results and are generally on the safe side. The use of the theory is illustrated by numerical examples.

The basic assumption of the yield-line theory is that a reinforced concrete slab, similar to a continuous beam or frame of a perfectly plastic material, will develop yield hinges under overload, but will not collapse until a mechanism is formed. The hinges in the slab must be long lines, along which the maximum resisting moment of the slab will tend to oppose rotation. The general crack pattern which these yield lines will form may be deduced logically from geometry, and sometimes must be obtained from model or full-scale tests. Once the general pattern is known, a specific crack pattern may be calculated for a particular support and loading condition using the principle of virtual work or force equilibrium. Sometimes the resulting equations are too complex for direct solution, and systems of successive approximations must be used.

An economic study of reinforcement is a problem in which we are interested. Steel may be saved in plastic design of slabs by the use of orthotropic reinforcement, i.e., with

different quantities of steel per foot of width in different directions. If the equilibrium of rigid portions in a mechanism is examined, then reductions in the total reinforcement can be made.

The yield-line theory seems best suited for ultimate load design. However, experience has indicated that a design based on flexural strength using the yield-line theory must be supplemented by a check of the conditions under service loads, particularly of deflections under sustained loading.